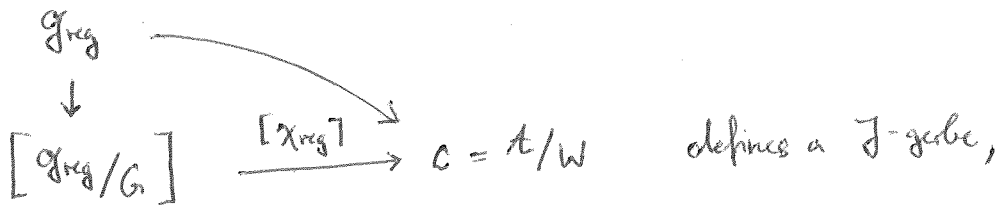


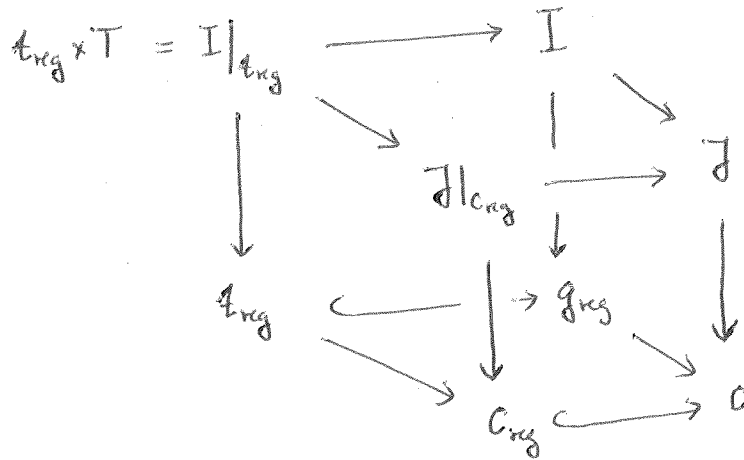
Recall:



where \mathcal{J} is the descent of the universal centralizer $I \subset g_{\text{reg}} \times G$
 \downarrow
 g_{reg}

To see what \mathcal{J} "is", look at $[\mathcal{A}_{\text{reg}}/N] \simeq [g_{\text{reg,ss}}/G]$.

Have



so $\mathcal{J}|_{\mathcal{C}_{\text{reg}}}(U \rightarrow \mathcal{C}_{\text{reg}}) = \{ f: U \times_{\mathcal{C}_{\text{reg}}} \mathcal{A}_{\text{reg}} \rightarrow T \}_{W\text{-equivariant}}$

$$= \pi_* (\mathcal{A}_{\text{reg}} \times T)^W(U),$$

in particular

$$B\mathcal{J}|_{\mathcal{C}_{\text{reg}}}(U) = \{ W\text{-equivariant principal bundles } P \text{ on the canonical cover } \tilde{U} = U \times_{\mathcal{C}_{\text{reg}}} \mathcal{A}_{\text{reg}} \}$$

How to extend over all of \mathcal{C} ?

Step 1: Construct morphism $\mathcal{J} \rightarrow \pi_* (\mathcal{A} \times T)$ for $\pi: \mathcal{A} \rightarrow \mathcal{C}$

Step 2: $\pi_* (\mathcal{A} \times T)$ representable by a smooth affine group scheme ("Weil restriction")

For step 1, define \tilde{g}_{reg} by the Cartesian diagram

$$\begin{array}{ccc} \tilde{g}_{\text{reg}} & \xrightarrow{\tilde{\chi}_{\text{reg}}} & \mathcal{A} \\ \downarrow & \square & \downarrow \\ g_{\text{reg}} & \xrightarrow{\chi_{\text{reg}}} & \mathcal{C} \end{array}$$

"Grothendieck-Springer resolution"

Fact: $\tilde{g}_{\text{reg}} = \{ (x, gB) \in g_{\text{reg}} \times G/B \mid x \in \text{Ad}(g)(\mathfrak{b}) \}$, $\mathfrak{b} := \text{Lie}(B)$
 and $\tilde{\chi}_{\text{reg}}: (x, gB) \mapsto \text{projection of } \text{Ad}(g)(x) \in \mathfrak{b}$

Adjunction \Rightarrow for step 1, suffices to construct $\pi^* \mathcal{J} \rightarrow \mathcal{A} \times T$ over \mathcal{A} .

\Rightarrow Will be enough to construct

$$\begin{array}{ccc} \tilde{\chi}_{\text{reg}}^* \pi^* \mathcal{J} & \longrightarrow & \tilde{\chi}_{\text{reg}}^* (\mathcal{A} \times T) \\ \parallel & & \parallel \\ \mathcal{J}|_{\tilde{g}_{\text{reg}}} & & \tilde{g}_{\text{reg}} \times T \end{array}$$

Have universal Borel gp scheme $\underline{B} \downarrow \tilde{g}_{\text{reg}}$

Fact (a) $x \in g_{\text{reg}} \cap \mathfrak{b} \Rightarrow I_x \subset \underline{B}$ (b) $\underline{B}/\Gamma_{\underline{B}, \underline{B}} \cong \tilde{g}_{\text{reg}} \times T$

So we get $\mathcal{J}|_{\tilde{g}_{\text{reg}}} \xrightarrow{\exists} \underline{B} \rightarrow \underline{B}/\Gamma_{\underline{B}, \underline{B}} \cong \tilde{g}_{\text{reg}} \times T$

hence we get $\mathcal{J} \xrightarrow{F} \pi_* (\mathcal{A} \times T)$.

Question What's the image of this morphism?

Note: F factors over $\pi_* (\mathcal{A} \times T)^w =: \bar{\mathcal{T}}$, since it does so over tr_{reg} & $\bar{\mathcal{T}}$ is also representable.

Prop (DG, sect. 12) $Z_G(x_n)^{ss} = Z_G$ for $x_n \in \mathfrak{g}_{reg}$.
 $\{t \in T^W \mid \alpha(t) = 1 \ \forall \text{ roots } \alpha\}$

This gives a candidate for $\text{Im}(J \rightarrow \bar{T}) =$

$$\mathcal{T}(U) := \left\{ f: U \times t \rightarrow W \text{-equivariant} \right. \\ \left. \begin{array}{l} \text{sth } \forall x \in U \times t \text{ and all } \alpha \text{ with } \alpha(x) = x, \\ \text{one has } \alpha(f(x)) = 1 \end{array} \right\}$$

Prop. $J \cong \mathcal{T}$.

Proof.

Step 1: $\mathcal{T} \subset \bar{T}$ represented by an open affine subscheme containing the conn. cpt \bar{T}^0 .

Step 2: $J \xrightarrow{\exists?} \mathcal{T}$ affine gp schemes / \mathbb{C}

$C_{\text{sing}} := \text{Image}(\text{double or worse ramification of } t \rightarrow c) \subset C$

$$\text{codim}(C_{\text{sing}}, C) = 2$$

\Rightarrow suffices to prove equality over $C \setminus C_{\text{sing}}$.

$$\begin{array}{ccc} 2.1 \text{ m} & J \rightarrow \bar{T} \text{ factors over } \mathcal{T} & \xleftrightarrow[\text{step 1}]{\iff} \pi_0(J_a) \rightarrow \pi_0(\bar{T}_a) \\ & & \begin{array}{c} \exists? \Rightarrow \\ \uparrow \\ \pi_0(\mathcal{T}_a) \end{array} \end{array}$$

$$\text{But } 0 \rightarrow Z_G \rightarrow J_a \rightarrow J_{a, \text{Ad}G} \rightarrow 0 \\ \downarrow \\ \text{fact: connected!}$$

$$\Rightarrow \pi_0(Z_G) \twoheadrightarrow \pi_0(J_a)$$

$$\Rightarrow \text{ok since } Z_G \subset \mathcal{T}_a$$

2.2

Prop. $\forall \alpha \in C \setminus C_{\text{sing}}$ with $\alpha \in \text{Im}(d^+ \hookrightarrow T \rightarrow C)$,

one has: (i) $C_{H_\alpha} \xrightarrow{i_\alpha} C$ is étale at some $a_\alpha \mapsto \alpha$,

where $\bullet H_\alpha = Z_G(\ker(d: T \rightarrow G_m)) \leftarrow \text{ss rank 1}$

$\bullet C_{H_\alpha} = \mathfrak{t}/W_\alpha$

(ii) $i_\alpha^* \mathcal{J} \cong \mathcal{J}_{H_\alpha}$ locally on C_{H_α}

$$\Rightarrow [\mathfrak{g}_{\text{reg}}/G] \times_{C_{H_\alpha}} \cong [\mathfrak{h}_\alpha/H_\alpha]$$

$$\& i_\alpha^* \mathcal{I} \cong \mathcal{I}_{H_\alpha}$$

$$(iii) H_\alpha \cong \begin{cases} \mathcal{O}(2) \\ \text{PGL}(2) \times \text{tors.} \\ \text{SL}(2) \end{cases}$$

\Rightarrow enough to prove $\mathcal{J} \cong \mathcal{I}$ for these three gps.

Proof of the prop.

$\alpha \in C \setminus C_{\text{sing}}$ $\Rightarrow \exists x_s \in T$ sth $d(x_s) = 0$
simple branch pt $p(x_s) \neq 0 \forall p \neq \alpha$

$$\text{Let } \mathfrak{h}_\alpha := \text{Lie}(H_\alpha) \\ = \mathfrak{t} \oplus X_\alpha \oplus X_{-\alpha}$$

Claim: x_n reg. nilpotent in $\mathfrak{h}_\alpha \Rightarrow x = x_{\text{ss}} + x_n \in \mathfrak{g}_{\text{reg}}$.

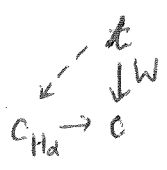
\hookrightarrow holds since $\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(x_n) \cap \underbrace{\mathfrak{z}_{\mathfrak{g}}(x_{\text{ss}})}_{=\mathfrak{h}_\alpha} = \mathfrak{z}_{\mathfrak{h}_\alpha}(x_n)$.

Put $U_\alpha = T \setminus \bigcup_{p \neq \alpha} A_p \subset T$ for $A_p = p^{-1} \subset T$

$U_\alpha/W := \text{image}(U_\alpha \hookrightarrow T \rightarrow C) \subset C$.

Claim: $U_\alpha/W_\alpha \rightarrow U_\alpha/W$ is étale
 \cap
 $C_{H_\alpha} \setminus C_{H_\alpha, \text{sing}} \quad \cap \quad C_{\text{sing}}$

Now $i_\alpha^* \mathcal{T} = \mathcal{T}_{H_\alpha}$:



On U_α , the only condition is on α
 \Rightarrow enough to check
 $i_\alpha^* \overline{\mathcal{T}}_{U_\alpha/W} \cong \mathcal{T}_{H_\alpha|_{U_\alpha/W_\alpha}}$

on U_α
 branching only given by α ,
 so W_α -equiv. maps extends to W -equiv. maps.

$$\begin{aligned}
 i_\alpha^* \overline{\mathcal{T}}(U \rightarrow C_{H_\alpha}) &= \text{Hom}_W(U \times_C \mathcal{T}, \mathcal{T}) \\
 \overline{\mathcal{T}}_{H_\alpha}(U \rightarrow C_{H_\alpha}) &= \text{Hom}_{W_\alpha}(U \times_{C_{H_\alpha}} \mathcal{T}, \mathcal{T})
 \end{aligned}$$

\rightsquigarrow ok over U_α , and over the U_β 's it's fine anyway ($\beta \neq \alpha$)

$a_\alpha \mapsto a$

$$J_{H_\alpha, a_\alpha} = Z_{H_\alpha}(x_n)$$

$$J_a = Z_G(x_{ss} + x_n) = \underbrace{Z_G(x_{ss})}_{H_\alpha} \cap Z_G(x_n)$$

$J_{H_\alpha} \hookrightarrow J$ closed subgp scheme. $\dots \Rightarrow$ prop. 2.2.

Last step: Check claim for $PGL(2)$
 $GL(2)$
 $SL(2)$.

Assume $J^0 \subset \mathcal{T}^0$ (prop. 12.5 in DG)

Then enough to check that connected components match. [...]

