Recall:

$$\begin{tikzcd} \mathfrak{g}_{reg} \arrow[d] \arrow[drr, bend left] & & \\ {[\mathfrak{g}_{reg}/G]} \arrow[rr, "{[\chi_{reg}]}"] & & \mathfrak{c} = \mathfrak{t}/W \end{tikzcd}$$ defines a J-gerbe,

where $\begin{matrix} J \\ \downarrow \\ \mathfrak{c} \end{matrix}$ is the descent of the universal centralizer $\begin{matrix} I \subset \mathfrak{g}_{reg} \times G \\ \downarrow \\ \mathfrak{g}_{reg} \end{matrix}$

To see what J "is", look at $[\mathfrak{t}_{reg}/N] \simeq [\mathfrak{g}_{reg,ss}/G]$.

Have

$$\begin{tikzcd} \mathfrak{t}_{reg} \times T = I|_{\mathfrak{t}_{reg}} \arrow[rr] \arrow[dr] \arrow[dd] & & I \arrow[dr] \arrow[dd] & \\ & J|_{\mathfrak{c}_{reg}} \arrow[rr] \arrow[dd] & & J \arrow[dd] \\ \mathfrak{t}_{reg} \arrow[rr, hook] \arrow[dr] & & \mathfrak{g}_{reg} \arrow[dr] & \\ & \mathfrak{c}_{reg} \arrow[rr, hook] & & \mathfrak{c} \end{tikzcd}$$

so
$$J|_{\mathfrak{c}_{reg}}(U \to \mathfrak{c}_{reg}) = \{ f : U \times_{\mathfrak{c}_{reg}} \mathfrak{t}_{reg} \to T \text{ } W\text{-equivariant} \}$$

$$= \pi_*(\mathfrak{t}_{reg} \times T)^W(U),$$

in particular
$$BJ|_{\mathfrak{c}_{reg}}(U) = \{ W\text{-equivariant ppal bdles } P \text{ on the cameral cover } \tilde{U} = U \times_{\mathfrak{c}_{reg}} \mathfrak{t}_{reg} \}.$$

How to extend over all of $\mathfrak{c}$?

Step 1: Construct morphism $J \to \pi_*(\mathfrak{t} \times T)$ for $\pi : \mathfrak{t} \to \mathfrak{c}$

Step 2: $\pi_*(\mathfrak{t} \times T)$ representable by a smooth affine group scheme ("Weil restriction")

For step 1, define $\tilde{\mathfrak{g}}_{reg}$ by the Cartesian diagram

$$\begin{tikzcd} \tilde{\mathfrak{g}}_{reg} \arrow[r, "\tilde{\chi}_{reg}"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{t} \arrow[d] \\ \mathfrak{g}_{reg} \arrow[r, "\chi_{reg}"'] & \mathfrak{c} \end{tikzcd}$$

"Grothendieck-Springer resolution"

<u>Fact</u>: $\tilde{\mathfrak{g}}_{reg} = \{ (x, gB) \in \mathfrak{g}_{reg} \times G/B \mid x \in Ad(g)(\mathfrak{b}) \}$, $\quad \mathfrak{b} := Lie(B)$

and $\tilde{\chi}_{reg} : (x, gB) \longmapsto$ projection of $Ad(g^{-1})(x) \in \mathfrak{b}$

Adjunction $\Longrightarrow$ for step 1, suffices to construct $\pi^* J \longrightarrow \mathfrak{t} \times T$ over $\mathfrak{t}$.

$\Longrightarrow$ Will be enough to construct

$$\begin{array}{ccc} \tilde{\chi}_{reg}^* \pi^* J & \longrightarrow & \tilde{\chi}_{reg}^*(\mathfrak{t} \times T) \\ \parallel & & \parallel \\ J|_{\tilde{\mathfrak{g}}_{reg}} & & \tilde{\mathfrak{g}}_{reg} \times T \end{array}$$

Have universal Borel gp scheme $\begin{array}{c} \underline{B} \\ \downarrow \\ \tilde{\mathfrak{g}}_{reg} \end{array}$

<u>Fact</u> (a) $x \in \mathfrak{g}_{reg} \cap \mathfrak{b} \Longrightarrow I_x \subset B$ $\quad$ (b) $\underline{B}/[\underline{B}, \underline{B}] \cong \tilde{\mathfrak{g}}_{reg} \times T$

So we get

$$\begin{tikzcd} J|_{\tilde{\mathfrak{g}}_{reg}} \arrow[r, hook, "\exists"] \arrow[rrr, bend right] & \underline{B} \arrow[r] & \underline{B}/[\underline{B}, \underline{B}] \arrow[r, phantom, "\cong"] & \tilde{\mathfrak{g}}_{reg} \times T \end{tikzcd}$$

hence we get $\quad J \xrightarrow{F} \pi_*(\mathfrak{t} \times T)$.

<u>Question</u> What's the image of this morphism?

<u>Note</u>: F factors over $\pi_*(\mathfrak{t} \times T)^W =: \overline{\mathcal{T}}$, since it does so over $\mathfrak{t}_{reg}$ & $\overline{\mathcal{T}}$ is also representable.

Take a branch pt $a \in \mathrm{Branch}(\mathfrak{t} \to \mathfrak{c})$.

Fix $x \mapsto a$, and take a reflection $s_\alpha \in W$ with $s_\alpha(x) = x$.

Then if $U \ni a$ & $f \in \mathcal{J}(U)$, we have

$$\begin{tikzcd} U \times_{\mathfrak{c}} \mathfrak{t} \arrow[rr, "f"] \arrow[dr, dashed, "\exists!"'] & & T \\ & T^{s_\alpha} \arrow[ur, hook] & \end{tikzcd}$$

in particular $\alpha(f(x)) \in \{\pm 1\}$

<u>Remark.</u> $\pi^* \mathcal{J} \longrightarrow \mathfrak{t} \times T$

$$G = Gl_2(\mathbb{C}) \rightsquigarrow Z_G\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \left\{ \begin{pmatrix} \lambda & z \\ 0 & \lambda \end{pmatrix} \,\middle|\, \begin{matrix} \lambda, z \in k \\ \lambda \neq 0 \end{matrix} \right\}$$

$$Z_G^{ss} = Z_G$$

$$\rightsquigarrow \lim(\mathcal{J}) \subset \tau \subset \overline{\tau}$$

where $\tau(U) := \{ f : U \times_{\mathfrak{c}} \mathfrak{t} \to T$
W-equivariant
with $\alpha(f(x)) \neq -1$
$\forall \alpha$ with $s_\alpha(x) = x \}$

<u>Higher rank:</u>

<u>Remarks</u> 1) $x \in \mathfrak{g}_{rg} \iff x_n \in \mathfrak{z}_{\mathfrak{g}}(x_{ss})_{reg}$

$x = x_n + x_s$ (nilpot. / semisimple) $\iff \dim \underbrace{\mathfrak{z}_{\mathfrak{g}}(x)}_{} = \mathrm{rk}\, G$

$$\mathfrak{z}_{\mathfrak{g}}(x) = \mathfrak{z}_{\mathfrak{g}}(x_s) \cap \mathfrak{z}_{\mathfrak{g}}(x_n) = \mathfrak{z}_{\mathfrak{z}_{\mathfrak{g}}(x_s)}(x_n)$$

2) $L := Z_G(x_s)$ reductive

$$\begin{tikzcd} \left[\mathfrak{z}_{\mathfrak{g}}(x_s) / Z_G(x_s) \right] \arrow[r] \arrow[d] & \left[\cdots \right] \arrow[d] \\ \mathfrak{c}_L \arrow[r] & \mathfrak{c} = \mathfrak{t}/W \end{tikzcd}$$

$\mathfrak{c}_L = \mathfrak{t}/W_L$

"everything pulls back nicely"

Prop (DG, sect. 12) $Z_G(x_n)^{ss} = Z_G$ for $x_n \in \mathfrak{g}^{reg}$.
$$Z_G = \{ t \in T^W \mid \alpha(t) = 1 \ \forall \text{ roots } \alpha \}$$

This gives a candidate for $\operatorname{Im}(J \longrightarrow \overline{\mathcal{T}})$:

$$\mathcal{T}(U) := \left\{ f : U \times_{\mathfrak{c}} \mathfrak{t} \to \ \text{W-equivariant sth } \forall x \in U \times_{\mathfrak{c}} \mathfrak{t} \text{ and all } \alpha \text{ with } s_\alpha(x) = x, \text{ one has } \alpha(f(x)) = 1 \right\}$$

Prop. $J \cong \mathcal{T}$.

Proof.

Step 1: $\mathcal{T} \subset \overline{\mathcal{T}}$ represented by an open affine subscheme containing the conn. cpt $\overline{\mathcal{T}}^0$.

Step 2: $J \xrightarrow{\exists ?} \mathcal{T}$ affine gp schemes $/\mathfrak{c}$

$\mathfrak{c}_{sing} :=$ Image (double or worse ramification of $\mathfrak{t} \to \mathfrak{c}$) $\subset \mathfrak{c}$

$\operatorname{codim}(\mathfrak{c}_{sing}, \mathfrak{c}) = 2$

$\Longrightarrow$ suffices to prove equality over $\mathfrak{c} \setminus \mathfrak{c}_{sing}$.

2.1 $\quad J \longrightarrow \overline{\mathcal{T}}$ factors over $\mathcal{T}$ $\overset{\text{step 1}}{\Longleftrightarrow}$

$$\begin{array}{ccc} \pi_0(J_a) & \longrightarrow & \pi_0(\overline{\mathcal{T}}_a) \\ & \overset{\exists ?}{\searrow} & \nearrow \\ & & \pi_0(\mathcal{T}_a) \end{array}$$

But $0 \to Z_G \to J_a \longrightarrow J_{a, \mathrm{Ad}G} \to 0$

$J_{a,\mathrm{Ad}G}$ fact: connected!

$\Longrightarrow \pi_0(Z_G) \twoheadrightarrow \pi_0(J_a)$

$\Longrightarrow$ ok since $Z_G \subset \mathcal{T}_a$.

2.2 **Prop.** $\forall a \in c \setminus c_{sing}$ with $a \in \mathrm{Im}(\alpha^\perp \hookrightarrow \mathfrak{t} \to c)$,

one has: (i) $c_{H_\alpha} \xrightarrow{i_\alpha} c$ is étale at some $a_\alpha \mapsto a$,

where • $H_\alpha = Z_G(\ker(\alpha : T \to \mathbb{G}_m))$ ← ss rank 1

• $c_{H_\alpha} = \mathfrak{t}/W_\alpha$

(ii) $i_\alpha^* J \cong J_{H_\alpha}$ locally on c_{H_α}

$(\Rightarrow [\mathfrak{g}_{reg}/G] \times c_{H_\alpha} \cong [\mathfrak{h}_\alpha / H_\alpha])$

& $i_\alpha^* \mathcal{T} \cong \mathcal{T}_{H_\alpha}$

(iii) $H_\alpha \cong \begin{cases} GL(2) \\ PGL(2) \\ SL(2) \end{cases} \times$ torus.

$\Rightarrow$ enough to prove $J \cong \mathcal{T}$ for these three gps.

Proof of the prop.

$a \in c \setminus c_{sing}$ $\Rightarrow$ $\exists\, x_s \in \mathfrak{t}$ sth $\alpha(x_s) = 0$
simple branch pt $\qquad \varphi(x_s) \neq 0 \ \forall \varphi \neq \alpha$

Let $\mathfrak{h}_\alpha := \mathrm{Lie}(H_\alpha)$.
$= \mathfrak{t} \oplus X_\alpha \oplus X_{-\alpha}$.

Claim: x_n reg. nilpotent in $\mathfrak{h}_\alpha \Rightarrow x = x_{ss} + x_n \in \mathfrak{g}_{reg}$.

↳ holds since $\mathfrak{z}_\mathfrak{g}(x) = \mathfrak{z}_\mathfrak{g}(x_n) \cap \underbrace{\mathfrak{z}_\mathfrak{g}(x_{ss})}_{=\mathfrak{h}_\alpha} = \mathfrak{z}_{\mathfrak{h}_\alpha}(x_n)$.

Put $U_\alpha = \mathfrak{t} \setminus \bigcup_{\varphi \neq \alpha} \mathfrak{t}_\varphi \subset \mathfrak{t}$ for $\mathfrak{t}_\varphi = \varphi^\perp \subset \mathfrak{t}$

$U_\alpha/W := \mathrm{image}(U_\alpha \hookrightarrow \mathfrak{t} \to c) \subset c$.

Claim:
$$\begin{array}{ccc} U_\alpha/W_\alpha & \longrightarrow & U_\alpha/W \\ \cap & & \cap \\ C_{H_\alpha}\setminus C_{H_\alpha,\mathrm{sing}} & & C\setminus C_{\mathrm{sing}} \end{array}$$
is étale

Now $i_\alpha^* \tau = \tau_{H_\alpha}$:

$$\begin{tikzcd} & \mathfrak{t} \arrow[dl, dashed] \arrow[d, "W"] \\ C_{H_\alpha} \arrow[r] & C \end{tikzcd}$$

On U_α, the only condition is on α
$\Rightarrow$ enough to check
$i_\alpha^* \bar\tau|_{U_\alpha/W} \cong \tau_{H_\alpha}|_{U_\alpha/W_\alpha}$.

$$i_\alpha^* \bar\tau (U \to C_{H_\alpha}) = \mathrm{Hom}_W(U \times_C \mathfrak{t}, T)$$

$$\bar\tau_{H_\alpha}(U \to C_{H_\alpha}) = \mathrm{Hom}_{W_\alpha}(U \times_{C_{H_\alpha}} \mathfrak{t}, T)$$

on U_α branching only given by α, so W_α-equiv. maps extends to W-equiv. maps.

$\rightsquigarrow$ ok over U_α, and over the U_β's it's fine anyway ($\beta \neq \alpha$)

$a_\alpha \mapsto a$

$J_{H_\alpha, a_\alpha} = Z_{H_\alpha}(x_n)$

$J_a = Z_G(x_{ss} + x_n) = \underbrace{Z_G(x_{ss})}_{H_\alpha} \cap Z_G(x_n)$

$J_{H_\alpha} \hookrightarrow J$ closed subgp scheme. ... $\Longrightarrow$ prop. 2.2.

Last step: Check claim for $PGl(2)$
$Gl(2)$
$Sl(2)$.

Assume $J^0 \subset T^0$ (prop. 12.5 in DG)
Then enough to check that connected components match. [...]

$\square$